Calculation of Photonuclear Process in the Region of Several Tens MeV

- Formulation of Exact Transition Rate for High Energy γ-Ray -

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The electromagnetic field approximated by using long wave-length limit is not valid for heavy nuclear mass or high energy γ-ray transition. To examine the contribution of the electric multipole field that is neglected in long wave-length limit, we formulize the El transition rate for the strict electric multipole field and compare quantitatively this result with Weisskopf estimate.

Ⅰ. Introduction

The long wave-length approximation is often used to estimate nuclear electric and magnetic transition for wide range of nuclear mass or γ-ray energy. However, the long wave-length approximation is not accurate in case of estimation of the transition rate for the heavy nuclei or high energy γ-ray. Therefore, in this work, we treat the electric multipole field strictly to formalize the El transition rate that is valid for heavy nuclei or high energy γ-ray. For simplification, we take the nuclear wave functions that are the same as Weisskopf estimation. We compare the transition rate with the Weisskopf unit in preliminary case.

Firstly, we briefly follow a derivation process of the El transition rate without the long-wavelength approximation in section Ⅱ. Next, we compare the obtained transition rate with the Weisskopf unit in section Ⅲ. Lastly, we summarize the result in section Ⅳ.

Ⅱ. Method

In this section, we survey formalization of the transition rate with the electric multipole field that is not approximated by using the long-wavelength.

We rewrite the El electric multipole field

$$\vec{A}_{l\mu}(\vec{x};E) \equiv \sqrt{\frac{l+1}{2l+1}}\, j_{l-1}\left(\left|\vec{k}\right| r\right)\vec{Y}_{l\mu,l-1}(\Omega) - \sqrt{\frac{l}{2l+1}}\, j_{l+1}\left(\left|\vec{k}\right| r\right)\vec{Y}_{l\mu,l+1}(\Omega) \tag{2.1}$$

into the two parts of the longitudinal multipole field

$$\vec{A}_{l\mu}(\vec{x};L) \equiv \sqrt{\frac{l}{2l+1}}\; j_{l-1}\left(\left|\vec{k}\right| r\right)\vec{Y}_{l\mu,l-1}(\Omega) + \sqrt{\frac{l+1}{2l+1}}\; j_{l+1}\left(\left|\vec{k}\right| r\right)\vec{Y}_{l\mu,l+1}(\Omega)$$

$$= \frac{1}{\left|\vec{k}\right|}\nabla\left[j_l\left(\left|\vec{k}\right| r\right)Y_{l\mu}(\Omega)\right] \tag{2.2}$$

and another term. Thus, the rewritten El electric multipole field is

$$\vec{A}_{l\mu}(\vec{x};E) = \sqrt{\frac{l+1}{l}}\frac{1}{\left|\vec{k}\right|}\nabla\left[j_l\left(\left|\vec{k}\right| r\right)Y_{l\mu}(\Omega)\right] - \sqrt{\frac{2l+1}{l}}\; j_{l+1}\left(\left|\vec{k}\right| r\right)\vec{Y}_{l\mu,l+1}(\Omega) \quad . \tag{2.3}$$

Where, $\vec{k}$ and $\mu = \pm 1$ are wave vector and polarization index of photon fields, respectively. $\vec{Y}_{l\mu,l\pm1}(\Omega)$ are vector spherical harmonic functions defined as

$$\vec{Y}_{l\mu,l\pm1}(\Omega) \equiv \sum_{\alpha=-(l+1)}^{l+1}\sum_{\beta=-1}^{1}\left((l\pm1)\,1\,l\left|\alpha\;\beta\;\mu\right.\right)Y_{(l+1)\,\alpha}(\Omega)\vec{e}_\beta \quad , \tag{2.4}$$

and $\left((l\pm1)\,1\,l\left|\alpha\;\beta\;\mu\right.\right)$ indicate the Clebsch-Gordan coefficients. In long-wavelength limit ($\left|\vec{k}\right| r << 1$), the electric multipole field is approximated as

$$\vec{A}_{l\mu}(\vec{x};E) = \sqrt{\frac{l+1}{l}}\frac{1}{\left|\vec{k}\right|}\nabla\left[\frac{\left(\left|\vec{k}\right| r\right)^l}{(2l+1)!!}Y_{l\mu}(\Omega)\right] \quad . \tag{2.3'}$$

The matrix element of the El transition that photon is emitted from excited nuclear state is

$$M_{fi}\left(\left|\vec{k}\right|,\mu;El\right) = -\sqrt{2\pi}(-i)^{l+1}\sqrt{2l+1}\int d^3x\,\langle f|\vec{j}(\vec{x})|i\rangle\cdot\vec{A}^*{}_{l\,\mu}(\vec{x};E) \tag{2.5}$$

$$= \sqrt{2\pi}(-i)^{l+1}\sqrt{2l+1}\left\{\sqrt{\frac{l+1}{l}}ic\int d^3x\,\langle f|\hat{\rho}(\vec{x})|i\rangle\, j_l\left(\left|\vec{k}\right| r\right)Y^*_{l\mu}(\Omega)\right.$$

$$\left. + \sqrt{\frac{2l+1}{l}}\int d^3x\,\langle f|\hat{\vec{j}}(\vec{x})|i\rangle\cdot j_{l+1}\left(\left|\vec{k}\right| r\right)\vec{Y}^*_{l\mu,l+1}(\Omega)\right\} \quad . \tag{2.6}$$

In viewpoint of one-particle model, the wave function of the nuclear state is given by the wave function of the nucleon. We take the same wave functions of the nuclear states as ones of Weisskopf estimation. The initial (ψ_i) and final state wave functions (ψ_f) of the one proton state are given as

$$\psi_i(r,\theta,\phi) = u(r)\, Y_{lm}(\theta,\phi) \tag{2.7}$$

and

$$\psi_f(r,\theta,\phi) = u(r)\, Y_{00}(\theta,\phi) = u(r)\frac{1}{\sqrt{4\pi}} \quad . \tag{2.8}$$

A radial wave function $u(r)$ is defined as

$$u(r) = \begin{cases} \sqrt{3/R^2} & , \ (\, 0 \le r \le R \,) \\ 0 & , \ (\, r > R \,) \end{cases} \quad . \tag{2.9}$$

Where, R indicate a nuclear radius. The expression of $u(r)$, Eq.(2.9), means that the probability of proton is uniform in the spherical nucleus and zero outside the nucleus. The excited nuclear state is characterized by spherical harmonic functions. The excited initial state has non-zero orbital angular momentum l and magnetic quantum number m. The quantum number l and m of the ground final state are equal to zero. For these wave functions, Eq.(2.7) and Eq.(2.8), we get the matrix element

$$M_{fi}\left(\left|\vec{k}\right|,\mu;El\right) = \frac{(-i)^l}{\sqrt{2}}\sqrt{\frac{(2l+1)(l+1)}{l}}\; ce_p\, \frac{3}{R^3}\left\{\delta_{\mu m}\int_0^R dr\, r^2\, j_l\left(\left|\vec{k}\right| r\right)\right.$$

$$\left. - \frac{\hbar}{2M_p c}\delta_{\mu m}\, l\int_0^R dr\, r\, j_{l+1}\left(\left|\vec{k}\right| r\right)\right\} \quad . \tag{2.10}$$

Where, the symbol M_p (e_p) indicate the mass (charge) of proton.

Averaging over m of initial state and summing over the two polarization of the photon, we get the El transition rate

$$T(l;E) = \frac{4\pi}{2l+1}\sum_{\mu=\pm1}\frac{1}{2l+1}\sum_{m=-l}^{l}\left[\frac{1}{4\pi}\frac{\left|\vec{k}\right|}{2\pi\hbar c^2}\left| M_{fi}\left(\left|\vec{k}\right|,\mu;El\right)\right|^2\right] \tag{2.11}$$

$$= \frac{e_p^{\,2}}{4\pi\hbar c}c\left|\vec{k}\right| 3^2\,\frac{2(l+1)}{l(2l+1)}2^{2l}\left(\left|\vec{k}\right| R\right)^{2l}\Bigg[$$

$$\sum_{n=0}^{\infty}\frac{1}{2n+l+3}\frac{(n+l)!}{(2n+2l+1)!}\frac{(-1)^n}{n!}\left(\left|\vec{k}\right| R\right)^{2n}\left(1-\frac{E_\gamma}{M_p c^2}\frac{l}{2(2n+2l+3)}\right)\Bigg]^2 \tag{2.12}$$

for the emission of unpolarized photon. The El transition rate is invariant with space rotation, and $4\pi/(2l+1)$ of Eq.(2.11) is a normalization factor for space rotation. If we substitute the electric multipole field Eq.(2.3)' for the matrix element Eq.(2.5), the transition rate Eq.(2.11) is equal to Weisskopf unit

$$T_{WU}(l;E) = \frac{e_p^{\ 2}}{4\pi\hbar c} c\left|\vec{k}\right| \left(\frac{3}{l+3} \right)^2 \frac{2(l+1)}{l(2l+1)[(2l+1)!!]^2} \left(\left|\vec{k}\right| R \right)^{2l} \quad . \qquad (2.12)'$$

III. Result

In this section, we estimate quantitatively the ratio of the El transition rate $T(l;E)$, Eq.(2.12), formularized in last section to Weisskopf unit $T_{WU}(l;E)$ Eq.(2.12)'. We consider only the $E1$, $E2$ and $E3$ transition rate in here.

From the ratio $T(l;E)/T_{WU}(l;E)$ with the nuclear mass number A (the γ ray energy E_γ) at fixed $E_\gamma = 60\text{MeV}$ ($A = 50$) [Fig.1,2], it is shown that the transition rate $T(l;E)$ is less than the Weisskopf unit $T_{WU}(l;E)$ and the ratio $T(l;E)/T_{WU}(l;E)$ decrease with increasing the mass number A or the γ ray energy E_γ. Moreover, the ratio $T(l;E)/T_{WU}(l;E)$ of the $E1$, $E2$ and $E3$ transition rate are nearly equal to 1 at $A \approx 0$ or $E_\gamma \approx 0\text{MeV}$, and split for the larger A or E_γ. These ratios satisfy the relation

$$\underset{\text{[E1]}}{\frac{T(l=1;E)}{T_{WU}(l=1;E)}} \le \underset{\text{[E2]}}{\frac{T(l=2;E)}{T_{WU}(l=2;E)}} \le \underset{\text{[E3]}}{\frac{T(l=3;E)}{T_{WU}(l=3;E)}} \quad . \qquad (3.1)$$

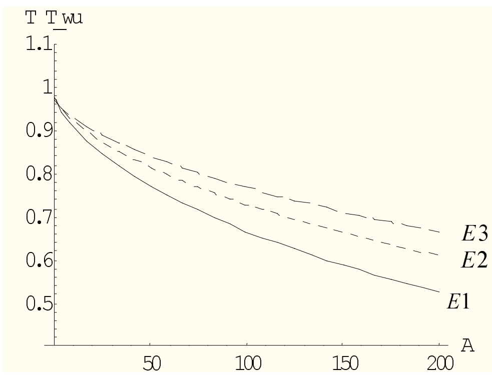

Fig.1 T/T_{WU} vs. mass number A at $E_\gamma = 60MeV$

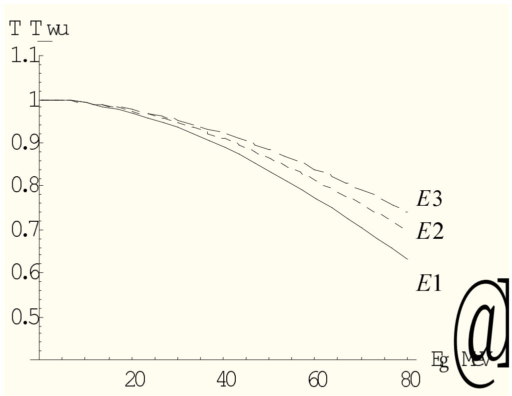

Fig.2 T/T_{WU} vs. γ energy E_γ at $A = 50$

We make table of the ratio $T(l;E)/T_{WU}(l;E)$ with the various combination of the nuclear mass number A and the γ ray energy E_γ [Table 1, 2, 3]. The above properties are valid for all A and E_γ also. As extreme example, the $T(l=1;E)/T_{WU}(l=1;E)$ of the $E1$ transition is 0.061 at $E_\gamma = 110\text{MeV}$ and $A=250$ ($|\vec{k}|R = 4.22$).

		A					
		10	50	100	150	200	250
Eγ [MeV]	10	0.996	0.991	0.987	0.984	0.981	0.978
	30	0.973	0.935	0.902	0.875	0.852	0.831
	50	0.934	0.835	0.755	0.693	0.642	0.598
	70	0.880	0.705	0.575	0.484	0.413	0.357
	90	0.813	0.560	0.396	0.293	0.222	0.171
	110	0.736	0.417	0.243	0.150	0.095	0.061

Table 1. $T(l=1;E)/T_{WU}(l=1;E)$ of $E1$ transition rate

		A					
		10	50	100	150	200	250
Eγ [MeV]	10	0.995	0.992	0.989	0.986	0.984	0.982
	30	0.975	0.946	0.921	0.900	0.881	0.865
	50	0.942	0.865	0.802	0.751	0.709	0.672
	70	0.898	0.759	0.652	0.573	0.510	0.458
	90	0.843	0.636	0.493	0.396	0.324	0.269
	110	0.780	0.510	0.344	0.245	0.180	0.134

Table 2. $T(l=2;E)/T_{WU}(l=2;E)$ of $E2$ transition rate

		A					
		10	50	100	150	200	250
Eγ [MeV]	10	0.995	0.992	0.990	0.988	0.986	0.984
	30	0.977	0.953	0.932	0.914	0.899	0.885
	50	0.948	0.884	0.831	0.788	0.752	0.720
	70	0.909	0.793	0.701	0.631	0.575	0.527
	90	0.861	0.686	0.558	0.468	0.400	0.345
	110	0.807	0.573	0.419	0.320	0.251	0.200

Table 3. $T(l=3;E)/T_{WU}(l=3;E)$ of $E3$ transition rate

Finally, we investigate the variation of the $T(l;E)/T_{WU}(l;E)$ with $|\vec{k}|R$ at $E_\gamma = 10\text{MeV}$ and $E_\gamma = 110\text{MeV}$ [Fig. 3, 4, 5]. The shape of the $T(l;E)/T_{WU}(l;E)$ are almost independent with γ ray energy.

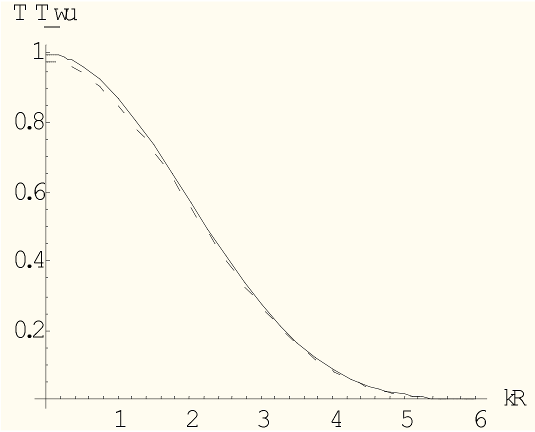

Fig.3 T/T_{WU} vs. $\left|\vec{k}\right|R$ in $E1$ transition

at $E_{\gamma} = 10\text{MeV}$ (a solid line)

and $E_{\gamma} = 110\text{MeV}$ (a broken line)

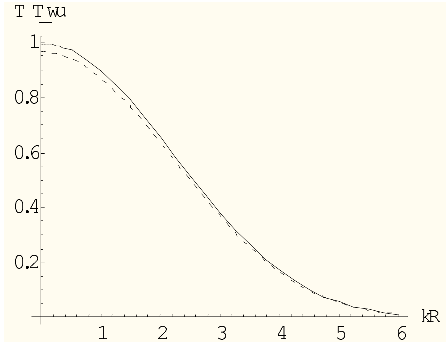

Fig.4 T/T_{WU} vs. $\left|\vec{k}\right|R$

in $E2$ transition

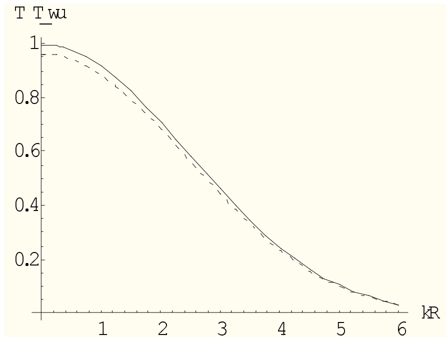

Fig.5 T/T_{WU} vs. $\left|\vec{k}\right|R$

in $E3$ transition

Ⅳ. Conclusion

We treated the electric multipole field strictly and formalized the El transition rate using the simple wave function that is used in the Weisskopf estimation. We estimated quantitatively the ratio of the transition rate $T(l;E)$ formulated by the strict electric multipole field to the Weisskopf unit $T_{WU}(l;E)$. In consequence, The ratio $T(l;E)/T_{WU}(l;E)$ is less than 1 and decrease with increasing the nuclear mass or γ-ray energy.

We did not consider the magnetic transition rate yet. But, it is possible to extend the electric transition to the magnetic transition with introducing the spin of the nucleon.

References

[1] V. F. Weisskopf, Phys. Rev. **83**, 1073 (1951)